

Generalized Rodriguez-Villegas supercongruences involving p -adic Gamma functions

Ji-Cai Liu

Department of Mathematics, East China Normal University, Shanghai 200241, China
jc2051@163.com

Abstract. Let $p \geq 5$ be a prime and $\langle a \rangle_p$ denote the least non-negative integer r with $a \equiv r \pmod{p}$. We mainly prove that for any p -adic integer a with $\langle a \rangle_p \equiv 0 \pmod{2}$

$$\sum_{k=0}^{p-1} \frac{\left(\frac{1}{2}\right)_k (-a)_k (a+1)_k}{(1)_k^3} \equiv (-1)^{\frac{p+1}{2}} \Gamma_p \left(-\frac{a}{2}\right)^2 \Gamma_p \left(\frac{a+1}{2}\right)^2 \pmod{p^2},$$

where $(x)_k = x(x+1) \cdots (x+k-1)$ and $\Gamma_p(\cdot)$ denotes the p -adic Gamma function. This partially extends four Rodriguez-Villegas supercongruences for truncated hypergeometric series ${}_3F_2$.

Keywords: Supercongruences; Truncated hypergeometric series; p -adic Gamma functions

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1 Introduction

Rodriguez-Villegas [12] observed the relationship between the number of points over F_p on hypergeometric Calabi-Yau manifolds and truncated hypergeometric series. Some interesting supercongruences for hypergeometric Calabi-Yau manifolds of dimension $D \leq 3$ were conjectured by Rodriguez-Villegas.

For complex numbers a_i, b_j and z , with none of the b_j being negative integers or zero, the truncated hypergeometric series are given by

$${}_rF_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; z \right]_n = \sum_{k=0}^n \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k} \cdot \frac{z^k}{k!},$$

where $(a)_0 = 1$ and $(a)_k = a(a+1) \cdots (a+k-1)$ for $k \geq 1$.

Throughout this paper, let $p \geq 5$ be a prime. Rodriguez-Villegas [12] posed four conjectural supercongruences for manifolds of dimension $D = 1$, which relate to certain elliptic curves.

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; 1 \right]_{p-1} &\equiv \left(\frac{-1}{p} \right) \pmod{p^2}, & {}_2F_1 \left[\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix}; 1 \right]_{p-1} &\equiv \left(\frac{-3}{p} \right) \pmod{p^2}, \\ {}_2F_1 \left[\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{matrix}; 1 \right]_{p-1} &\equiv \left(\frac{-2}{p} \right) \pmod{p^2}, & {}_2F_1 \left[\begin{matrix} \frac{1}{6}, \frac{5}{6} \\ 1 \end{matrix}; 1 \right]_{p-1} &\equiv \left(\frac{-1}{p} \right) \pmod{p^2}, \end{aligned}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Using the Gross-Koblitz formula, Mortenson [9,10] first proved these four supercongruences. Some extensions of these supercongruences to the case modulus p^3 were obtained by Z.-W. Sun [17] and Z.-H. Sun [16].

For manifolds of dimension $D = 2$, associated to certain modular K3 surfaces, another four supercongruences were conjectured. These were all of the form

$${}_3F_2\left[\begin{matrix} \frac{1}{2}, -a, a+1 \\ 1, 1 \end{matrix}; 1\right]_{p-1} \equiv c_p \pmod{p^2}, \quad (1.1)$$

where $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ and c_p is the p -th Fourier coefficient of a weight three modular form on a congruence subgroup of $SL(2, \mathbb{Z})$. The case when $a = -\frac{1}{2}$ was originally conjectured by Beukers and Stienstra [3], and confirmed by van Hamme [21], Ishikawa [7] and Ahlgren [1]. The other cases when $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ were partially proved by Mortenson [11], and finally proved by Z.-W. Sun [18].

Recently, Long and Ramakrishna [8, Theorem 3] obtained the following amazing supercongruence:

$${}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1\right]_{p-1} \equiv \begin{cases} -\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{p^2}{16}\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where $\Gamma_p(\cdot)$ denotes the p -adic Gamma function recalled in the next section.

Let $\langle a \rangle_p$ denote the least non-negative integer r with $a \equiv r \pmod{p}$. For any p -adic integer a with $\langle a \rangle_p \equiv 1 \pmod{2}$, Z.-H. Sun [15, Theorem 2.5] partially extended (1.1) as follows

$${}_3F_2\left[\begin{matrix} \frac{1}{2}, -a, a+1 \\ 1, 1 \end{matrix}; 1\right]_{p-1} \equiv 0 \pmod{p^2}. \quad (1.2)$$

Guo and Zeng [6, Theorem 1.3] have obtained an interesting q -analogue of (1.2). Using the same idea, Z.-H. Sun [15, Corollary 2.2] also showed that for $\langle a \rangle_p \equiv 1 \pmod{2}$

$${}_2F_1\left[\begin{matrix} -a, a+1 \\ 1 \end{matrix}; \frac{1}{2}\right]_{p-1} \equiv 0 \pmod{p^2}. \quad (1.3)$$

The cases when $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in (1.3) were dealt with by Z.-H. Sun [14,15], Z.-W. Sun [17,19] and Tauraso [20].

In this paper, we aim to determine the case $\langle a \rangle_p \equiv 0 \pmod{2}$ on the left-hand sides of (1.2) and (1.3). Our proof is based on some properties of the Morita's p -adic Gamma function and some combinatorial identities involving harmonic numbers.

Theorem 1.1 *Let $p \geq 5$ be a prime. For any p -adic integer a with $\langle a \rangle_p \equiv 0 \pmod{2}$, we have*

$${}_2F_1\left[\begin{matrix} -a, a+1 \\ 1 \end{matrix}; \frac{1}{2}\right]_{p-1} \equiv (-1)^{\frac{p+1}{2}} \Gamma_p\left(\frac{1}{2}\right) \Gamma_p\left(-\frac{a}{2}\right) \Gamma_p\left(\frac{a+1}{2}\right) \pmod{p^2}. \quad (1.4)$$

Theorem 1.2 *Let $p \geq 5$ be a prime. For any p -adic integer a with $\langle a \rangle_p \equiv 0 \pmod{2}$, we have*

$${}_3F_2 \left[\begin{matrix} \frac{1}{2}, -a, a+1 \\ 1, 1 \end{matrix}; 1 \right]_{p-1} \equiv (-1)^{\frac{p+1}{2}} \Gamma_p \left(-\frac{a}{2} \right)^2 \Gamma_p \left(\frac{a+1}{2} \right)^2 \pmod{p^2}. \quad (1.5)$$

In order to prove Theorem 1.2, we establish the following supercongruence which connects the left-hand side of (1.5) to that of (1.4).

Theorem 1.3 *Suppose $p \geq 5$ is a prime. For any p -adic integer a , we have*

$${}_3F_2 \left[\begin{matrix} \frac{1}{2}, -a, a+1 \\ 1, 1 \end{matrix}; 1 \right]_{p-1} \equiv {}_2F_1 \left[\begin{matrix} -a, a+1 \\ 1 \end{matrix}; \frac{1}{2} \right]_{p-1}^2 \pmod{p^2}. \quad (1.6)$$

Supercongruence (1.6) is a p -adic analogue of the following identity when $x = 1$:

$${}_3F_2 \left[\begin{matrix} \frac{1}{2}, -a, a+1 \\ 1, 1 \end{matrix}; x \right] = {}_2F_1 \left[\begin{matrix} -a, a+1 \\ 1 \end{matrix}; \frac{1 \pm \sqrt{1-x}}{2} \right]^2, \quad (1.7)$$

which can be deduced from the Clausen formula and a Pfaff transformation. See equation [4, (3.3)] for a derivation of this formula.

The organization of this paper is as follows. In the next section, we recall some properties of the p -adic Gamma function, and establish some combinatorial identities involving harmonic numbers. We prove Theorem 1.1 in Section 3 and prove Theorem 1.2 and 1.3 in the last section.

2 Some lemmas

We first recall some basic properties of the Morita's p -adic Gamma function. For more details, one can refer to [5, §11.6]. Let p be an odd prime and \mathbb{Z}_p denote the set of all p -adic integers. For $x \in \mathbb{Z}_p$, the Morita's p -adic Gamma function [5, Definition 11.6.5] is defined as

$$\Gamma_p(x) = \lim_{m \rightarrow x} (-1)^m \prod_{\substack{0 \leq k < m \\ (k, p) = 1}} k,$$

where the limit is for m tending to x p -adically in $\mathbb{Z}_{\geq 0}$.

Lemma 2.1 *Suppose p is an odd prime and $x \in \mathbb{Z}_p$. Then*

$$\Gamma_p(1) = -1, \quad (2.1)$$

$$\Gamma_p(x) \Gamma_p(1-x) = (-1)^{s_p(x)}, \quad (2.2)$$

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } |x|_p = 1, \\ -1 & \text{if } |x|_p < 1, \end{cases} \quad (2.3)$$

where $s_p(x) \in \{1, 2, \dots, p\}$ with $s_p(x) \equiv x \pmod{p}$ and $|\cdot|_p$ denotes the p -adic norm.

Lemma 2.2 (Long and Ramakrishna [8, Lemma 17, (4)]) Let p be an odd prime. If $a \in \mathbb{Z}_p, n \in \mathbb{N}$ such that none of $a, a+1, \dots, a+n-1$ in $p\mathbb{Z}_p$, then

$$(a)_n = (-1)^n \frac{\Gamma_p(a+n)}{\Gamma_p(a)}. \quad (2.4)$$

The following lemma is a special case of the theorem due to Long and Ramakrishna [8, Theorem 14].

Lemma 2.3 (Long and Ramakrishna [8, Theorem 14]) Suppose $p \geq 5$ is a prime. For any $a, b \in \mathbb{Z}_p$, we have

$$\Gamma_p(a+bp) \equiv \Gamma_p(a) (1 + G_1(a)bp) \pmod{p^2}, \quad (2.5)$$

where $G_1(a) = \Gamma'_p(a)/\Gamma_p(a) \in \mathbb{Z}_p$.

Lemma 2.4 Let p be an odd prime. For any $x \in \mathbb{Z}_p$, we have

$$G_1(x) \equiv G_1(1) + H_{s_p(x)-1} \pmod{p}, \quad (2.6)$$

where H_n denotes the n -th harmonic number $H_n = \sum_{k=1}^n \frac{1}{k}$.

Proof. The p -adic logarithm is defined as

$$\log_p(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n},$$

which converges for $x \in \mathbb{C}_p$ with $|x|_p < 1$. Taking the \log_p derivative on both sides of (2.3) gives

$$G_1(x+1) - G_1(x) = \begin{cases} \frac{1}{x} & \text{if } |x|_p = 1, \\ 0 & \text{if } |x|_p < 1. \end{cases} \quad (2.7)$$

For any p -adic integer a and b with $a \equiv b \pmod{p}$, we have $\Gamma_p(a) \equiv \Gamma_p(b) \pmod{p}$ and $\Gamma'_p(a) \equiv \Gamma'_p(b) \pmod{p}$, and so $G_1(a) \equiv G_1(b) \pmod{p}$. Repeatedly applying (2.7), we obtain

$$\begin{aligned} G_1(x) &\equiv G_1(s_p(x)) \pmod{p} \\ &= G_1(s_p(x) - 1) + \frac{1}{s_p(x) - 1} \\ &= G_1(1) + H_{s_p(x)-1}. \end{aligned}$$

This completes the proof. □

We also need some combinatorial identities involving harmonic numbers.

Lemma 2.5 *For any even non-negative integer n , we have*

$$\sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} \left(-\frac{1}{2}\right)^k = \frac{\binom{n}{n/2}}{(-4)^{n/2}}, \quad (2.8)$$

$$\sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(-\frac{1}{4}\right)^k = \frac{\binom{n}{n/2}^2}{4^n}, \quad (2.9)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} \left(-\frac{1}{2}\right)^k H_{n+k} = \frac{\binom{n}{n/2}}{(-4)^{n/2}} \left(\frac{3}{2}H_n - \frac{1}{2}H_{\frac{n}{2}}\right), \quad (2.10)$$

$$\sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(-\frac{1}{4}\right)^k H_{n+k} = \frac{\binom{n}{n/2}^2}{4^n} \left(\frac{5}{2}H_n - H_{\frac{n}{2}}\right). \quad (2.11)$$

Proof. Note the following identity [2, (2), pp.11]:

$${}_2F_1\left[\begin{matrix} a, b \\ \frac{a+b+1}{2} \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{b+1}{2}\right)} \quad \text{for } a+b=1. \quad (2.12)$$

Letting $a = -n, b = n+1$ in (2.12) and then noting that

$$\frac{(-n)_k (n+1)_k}{(1)_k^2} = (-1)^k \binom{2k}{k} \binom{n+k}{2k}, \quad (2.13)$$

we conclude that for even non-negative integer n

$$\sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} \left(-\frac{1}{2}\right)^k = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{-n+1}{2}\right) \Gamma\left(\frac{n+2}{2}\right)}.$$

Using the fact that $(x)_k = \Gamma(x+k)/\Gamma(x)$ and

$$\frac{\left(\frac{1}{2}\right)_k}{(1)_k} = \frac{\binom{2k}{k}}{4^k}, \quad (2.14)$$

we obtain

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{-n+1}{2}\right) \Gamma\left(\frac{n+2}{2}\right)} = \frac{\left(\frac{-n+1}{2}\right)_{\frac{n}{2}}}{(1)_{\frac{n}{2}}} = (-1)^{\frac{n}{2}} \frac{\left(\frac{1}{2}\right)_{\frac{n}{2}}}{(1)_{\frac{n}{2}}} = \frac{\binom{n}{n/2}}{(-4)^{n/2}}.$$

This completes the proof of (2.8).

Letting $a = n$ and $x = 1$ in (1.7) and then noting (2.13) and (2.14), we get

$$\sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(-\frac{1}{4}\right)^k = \left(\sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} \left(-\frac{1}{2}\right)^k\right)^2.$$

Then the proof of (2.9) follows from the above equation and (2.8).

Let A_n and B_n denote the numbers

$$A_n = \sum_{k=0}^{2n} \binom{2k}{k} \binom{2n+k}{2k} \left(-\frac{1}{2}\right)^k H_{2n+k},$$

$$B_n = \sum_{k=0}^{2n} \binom{2k}{k}^2 \binom{2n+k}{2k} \left(-\frac{1}{4}\right)^k H_{2n+k},$$

respectively. Using the software package developed by Schneider [13], we find that A_n and B_n satisfy the following recurrences:

$$4(n+2)^2(8n+7)A_{n+2} + (64n^3 + 248n^2 + 300n + 114)A_{n+1} + (2n+1)^2(8n+15)A_n = 0, \quad (2.15)$$

and

$$16(n+1)(n+2)^3(12n+11)B_{n+2} - 4(n+1)(2n+3)(48n^3 + 188n^2 + 230n + 89)B_{n+1} + (2n+1)^3(2n+3)(12n+23)B_n = 0. \quad (2.16)$$

It is not hard to verify that the right-hand sides of (2.10) and (2.11) also satisfy the recurrences (2.15) and (2.16), respectively. This completes the proof of (2.10) and (2.11). \square

It follows from (2.8)-(2.11) that for even non-negative integer n

$$\sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} \left(-\frac{1}{2}\right)^k \sum_{i=1}^k \frac{1}{n+i} = \frac{\binom{n}{n/2}}{(-4)^{n/2}} \left(\frac{1}{2}H_n - \frac{1}{2}H_{\frac{n}{2}}\right), \quad (2.17)$$

$$\sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(-\frac{1}{4}\right)^k \sum_{i=1}^k \frac{1}{n+i} = \frac{\binom{n}{n/2}^2}{4^n} \left(\frac{3}{2}H_n - H_{\frac{n}{2}}\right). \quad (2.18)$$

3 Proof of Theorem 1.1

By (2.13), we can rewrite (1.4) as

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{a+k}{2k} \left(-\frac{1}{2}\right)^k \equiv (-1)^{\frac{p+1}{2}} \Gamma_p\left(\frac{1}{2}\right) \Gamma_p\left(-\frac{a}{2}\right) \Gamma_p\left(\frac{a+1}{2}\right) \pmod{p^2}. \quad (3.1)$$

Let δ denote the number $\delta = (a - \langle a \rangle_p)/p$. It is clear that δ is a p -adic integer and $a = \langle a \rangle_p + \delta p$. Note that

$$\begin{aligned}
\binom{2k}{k} \binom{a+k}{2k} &= \binom{2k}{k} \binom{\langle a \rangle_p + \delta p + k}{2k} \\
&= \binom{2k}{k} \prod_{i=1}^k (\langle a \rangle_p + \delta p + i) \prod_{i=1}^k (\langle a \rangle_p + \delta p + 1 - i) \prod_{i=1}^{2k} i^{-1} \\
&\equiv \binom{2k}{k} \binom{\langle a \rangle_p + k}{2k} \left(1 + \delta p \left(\sum_{i=1}^k \frac{1}{\langle a \rangle_p + i} + \sum_{i=1}^k \frac{1}{\langle a \rangle_p + 1 - i} \right) \right) \pmod{p^2},
\end{aligned} \tag{3.2}$$

where we have used the fact that $\binom{2k}{k} \prod_{i=1}^{2k} i^{-1} \in \mathbb{Z}_p$ for $0 \leq k \leq p-1$. It follows that

$$\begin{aligned}
\text{LHS (3.1)} &\equiv \sum_{k=0}^{p-1} \binom{2k}{k} \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{2} \right)^k \\
&\quad \times \left(1 + \delta p \left(\sum_{i=1}^k \frac{1}{\langle a \rangle_p + i} + \sum_{i=1}^k \frac{1}{\langle a \rangle_p + 1 - i} \right) \right) \pmod{p^2}.
\end{aligned} \tag{3.3}$$

Let $b = p - \langle a \rangle_p$. It is clear that $\langle a \rangle_p \equiv -b \pmod{p}$ and $0 \leq b-1 \leq p-1$ is an even integer. By (2.17), we have

$$\begin{aligned}
&\sum_{k=0}^{p-1} \binom{2k}{k} \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{2} \right)^k \sum_{i=1}^k \frac{1}{\langle a \rangle_p + 1 - i} \\
&\equiv - \sum_{k=0}^{p-1} \binom{2k}{k} \binom{-b+k}{2k} \left(-\frac{1}{2} \right)^k \sum_{i=1}^k \frac{1}{b-1+i} \pmod{p} \\
&= - \sum_{k=0}^{p-1} \binom{2k}{k} \binom{b-1+k}{2k} \left(-\frac{1}{2} \right)^k \sum_{i=1}^k \frac{1}{b-1+i} \\
&= \frac{\binom{b-1}{(b-1)/2}}{(-4)^{(b-1)/2}} \left(\frac{1}{2} H_{\frac{b-1}{2}} - \frac{1}{2} H_{b-1} \right),
\end{aligned} \tag{3.4}$$

where we have utilized the fact that $\binom{-b+k}{2k} = \binom{b-1+k}{2k}$ in the second step. Using (2.8), we

obtain

$$\begin{aligned}
\frac{\binom{b-1}{(b-1)/2}}{(-4)^{(b-1)/2}} &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{b-1+k}{2k} \left(-\frac{1}{2}\right)^k \\
&\equiv \sum_{k=0}^{p-1} \binom{2k}{k} \binom{-\langle a \rangle_p - 1 + k}{2k} \left(-\frac{1}{2}\right)^k \pmod{p} \\
&= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{2}\right)^k \\
&= \frac{\binom{\langle a \rangle_p}{\langle a \rangle_p/2}}{(-4)^{\langle a \rangle_p/2}}. \tag{3.5}
\end{aligned}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}
&\sum_{k=0}^{p-1} \binom{2k}{k} \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{2}\right)^k \sum_{i=1}^k \frac{1}{\langle a \rangle_p + 1 - i} \\
&\equiv \frac{\binom{\langle a \rangle_p}{\langle a \rangle_p/2}}{(-4)^{\langle a \rangle_p/2}} \left(\frac{1}{2} H_{\frac{p-\langle a \rangle_p-1}{2}} - \frac{1}{2} H_{p-\langle a \rangle_p-1} \right) \pmod{p}. \tag{3.6}
\end{aligned}$$

Furthermore, using (2.8) and (2.17) we obtain

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{2}\right)^k = \frac{\binom{\langle a \rangle_p}{\langle a \rangle_p/2}}{(-4)^{\langle a \rangle_p/2}}, \tag{3.7}$$

and

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{2}\right)^k \sum_{i=1}^k \frac{1}{\langle a \rangle_p + i} = \frac{\binom{\langle a \rangle_p}{\langle a \rangle_p/2}}{(-4)^{\langle a \rangle_p/2}} \left(\frac{1}{2} H_{\langle a \rangle_p} - \frac{1}{2} H_{\frac{\langle a \rangle_p}{2}} \right). \tag{3.8}$$

Finally, combining (3.3) and (3.6)-(3.8) gives

$$\text{LHS (3.1)} \equiv \left(-\frac{1}{4}\right)^{\langle a \rangle_p/2} \binom{\langle a \rangle_p}{\langle a \rangle_p/2} \left(1 + \frac{\delta p}{2} \left(H_{\frac{p-\langle a \rangle_p-1}{2}} - H_{\frac{\langle a \rangle_p}{2}}\right)\right) \pmod{p^2}, \tag{3.9}$$

where we have used the fact $H_{p-1-k} \equiv H_k \pmod{p}$ for $0 \leq k \leq p-1$.

By (2.4) and (2.14), we have

$$\begin{aligned}
\left(-\frac{1}{4}\right)^{\langle a \rangle_p/2} \binom{\langle a \rangle_p}{\langle a \rangle_p/2} &= (-1)^{\langle a \rangle_p/2} \frac{\left(\frac{1}{2}\right)^{\langle a \rangle_p/2}}{(1)^{\langle a \rangle_p/2}} \\
&= (-1)^{\langle a \rangle_p/2} \frac{\Gamma_p(1) \Gamma_p\left(\frac{1+\langle a \rangle_p}{2}\right)}{\Gamma_p\left(\frac{1}{2}\right) \Gamma_p\left(1 + \frac{\langle a \rangle_p}{2}\right)}. \tag{3.10}
\end{aligned}$$

Applying (2.2), we obtain

$$\Gamma_p \left(\frac{1}{2} \right)^2 = (-1)^{\frac{p+1}{2}}, \quad (3.11)$$

$$\Gamma_p \left(1 + \frac{\langle a \rangle_p}{2} \right) \Gamma_p \left(-\frac{\langle a \rangle_p}{2} \right) = (-1)^{1+\langle a \rangle_p/2}. \quad (3.12)$$

Substituting (2.1), (3.11) and (3.12) into (3.10), and then using $\langle a \rangle_p = a - \delta p$, we get

$$\begin{aligned} \text{LHS (3.10)} &= (-1)^{\frac{p+1}{2}} \Gamma_p \left(\frac{1}{2} \right) \Gamma_p \left(\frac{1 + \langle a \rangle_p}{2} \right) \Gamma_p \left(-\frac{\langle a \rangle_p}{2} \right) \\ &= (-1)^{\frac{p+1}{2}} \Gamma_p \left(\frac{1}{2} \right) \Gamma_p \left(\frac{1 + a - \delta p}{2} \right) \Gamma_p \left(\frac{-a + \delta p}{2} \right). \end{aligned} \quad (3.13)$$

Applying (2.5) to the right-hand side of (3.13), we obtain

$$\begin{aligned} \text{LHS (3.10)} &\equiv (-1)^{\frac{p+1}{2}} \Gamma_p \left(\frac{1}{2} \right) \Gamma_p \left(\frac{1+a}{2} \right) \Gamma_p \left(-\frac{a}{2} \right) \\ &\quad \times \left(1 + \frac{\delta p}{2} \left(G_1 \left(-\frac{a}{2} \right) - G_1 \left(\frac{1+a}{2} \right) \right) \right) \pmod{p^2}. \end{aligned} \quad (3.14)$$

It follows from (3.9) and (3.14) that

$$\begin{aligned} \text{LHS (3.1)} &\equiv (-1)^{\frac{p+1}{2}} \Gamma_p \left(\frac{1}{2} \right) \Gamma_p \left(\frac{1+a}{2} \right) \Gamma_p \left(-\frac{a}{2} \right) \\ &\quad \times \left(1 + \frac{\delta p}{2} \left(H_{\frac{p-\langle a \rangle_{p-1}}{2}} - H_{\frac{\langle a \rangle_p}{2}} + G_1 \left(-\frac{a}{2} \right) - G_1 \left(\frac{1+a}{2} \right) \right) \right) \pmod{p^2}. \end{aligned}$$

In order to prove (3.1), it suffices to show that

$$H_{\frac{p-\langle a \rangle_{p-1}}{2}} - H_{\frac{\langle a \rangle_p}{2}} + G_1 \left(-\frac{a}{2} \right) - G_1 \left(\frac{1+a}{2} \right) \equiv 0 \pmod{p}. \quad (3.15)$$

By (2.6), we get

$$G_1 \left(-\frac{a}{2} \right) - G_1 \left(\frac{1+a}{2} \right) \equiv H_{s_p(-\frac{a}{2})-1} - H_{s_p(\frac{1+a}{2})-1} \pmod{p}. \quad (3.16)$$

Since $\langle a \rangle_p$ is an even integer, we have

$$s_p \left(-\frac{a}{2} \right) - 1 = p - \frac{\langle a \rangle_p}{2} - 1, \quad (3.17)$$

$$s_p \left(\frac{1+a}{2} \right) - 1 = \frac{p + \langle a \rangle_p + 1}{2} - 1. \quad (3.18)$$

Substituting (3.16)-(3.18) into the left-hand side of (3.15) gives

$$\text{LHS (3.15)} \equiv H_{\frac{p-\langle a \rangle_{p-1}}{2}} - H_{\frac{\langle a \rangle_p}{2}} + H_{p-\frac{\langle a \rangle_p}{2}-1} - H_{\frac{p+\langle a \rangle_p}{2}-1} \equiv 0 \pmod{p},$$

where we have utilized the fact that $H_{p-k-1} \equiv H_k \pmod{p}$ for $0 \leq k \leq p-1$.

4 Proof of Theorem 1.2 and 1.3

Proof of (1.5). The proof of (1.5) directly follows from (1.4), (1.6) and (3.11). \square

Proof of (1.6). By (1.2) and (1.3), (1.6) clearly holds when $\langle a \rangle_p \equiv 1 \pmod{2}$.

When $\langle a \rangle_p \equiv 0 \pmod{2}$, by (2.13), (2.14) and (3.9), supercongruence (1.6) is equivalent to

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{a+k}{2k} \left(-\frac{1}{4}\right)^k \\ & \equiv \left(\frac{1}{4}\right)^{\langle a \rangle_p} \left(\frac{\langle a \rangle_p}{\langle a \rangle_p/2}\right)^2 \left(1 + \delta p \left(H_{\frac{p-\langle a \rangle_p-1}{2}} - H_{\frac{\langle a \rangle_p}{2}}\right)\right) \pmod{p^2}. \end{aligned} \quad (4.1)$$

Applying (3.2) to the left-hand side of (4.1) yields

$$\begin{aligned} \text{LHS (4.1)} & \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{4}\right)^k \\ & \quad \times \left(1 + \delta p \left(\sum_{i=1}^k \frac{1}{\langle a \rangle_p + i} + \sum_{i=1}^k \frac{1}{\langle a \rangle_p + 1 - i}\right)\right) \pmod{p^2}. \end{aligned} \quad (4.2)$$

Using the same idea in the previous section and the identities (2.9) and (2.18), we can show that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{4}\right)^k \sum_{i=1}^k \frac{1}{\langle a \rangle_p + 1 - i} \\ & \equiv \frac{\left(\frac{\langle a \rangle_p}{\langle a \rangle_p/2}\right)^2}{4^{\langle a \rangle_p}} \left(H_{\frac{p-\langle a \rangle_p-1}{2}} - \frac{3}{2}H_{p-\langle a \rangle_p-1}\right) \pmod{p}, \end{aligned} \quad (4.3)$$

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{4}\right)^k = \frac{\left(\frac{\langle a \rangle_p}{\langle a \rangle_p/2}\right)^2}{4^{\langle a \rangle_p}}, \quad (4.4)$$

and

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{\langle a \rangle_p + k}{2k} \left(-\frac{1}{4}\right)^k \sum_{i=1}^k \frac{1}{\langle a \rangle_p + i} = \frac{\left(\frac{\langle a \rangle_p}{\langle a \rangle_p/2}\right)^2}{4^{\langle a \rangle_p}} \left(\frac{3}{2}H_{\langle a \rangle_p} - H_{\frac{\langle a \rangle_p}{2}}\right). \quad (4.5)$$

Combining (4.2)-(4.5), we obtain

$$\text{LHS (4.1)} \equiv \left(\frac{1}{4}\right)^{\langle a \rangle_p} \left(\frac{\langle a \rangle_p}{\langle a \rangle_p/2}\right)^2 \left(1 + \delta p \left(H_{\frac{p-\langle a \rangle_p-1}{2}} - H_{\frac{\langle a \rangle_p}{2}}\right)\right) \pmod{p^2}.$$

This completes the proof of (1.6). \square

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